Robin's Theorem, Primes, and a new elementary reformulation of the Riemann Hypothesis

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ROBIN'S THEOREM, PRIMES, AND A NEW ELEMENTARY REFORMULATION OF THE RIEMANN HYPOTHESIS

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Abstract

Let

$$G(n) = \frac{\sigma(n)}{n \log\log n} \qquad (n > 1),$$

where $\sigma(n)$ is the sum of the divisors of n. We prove that the Riemann Hypothesis is true if and only if 4 is the only composite number N satisfying

$$G(N) \geq \max\left(G(N/p), G(aN)\right),$$

for all prime factors p of N and each positive integer a. The proof uses Robin's and Gronwall's theorems on $G(n)$. An alternate proof of one step depends on two properties of superabundant numbers proved using Alaoglu and Erdős's results.

1. Introduction

The *sum-of-divisors function* σ is defined by

$$\sigma(n) := \sum_{d|n} d.$$

For example, $\sigma(4) = 7$ and $\sigma(pn) = (p+1)\sigma(n)$, if p is a prime not dividing n.

In 1913, the Swedish mathematician Thomas Gronwall [4] found the maximal order of σ.

Theorem 1 (Gronwall)**.** *The function*

$$G(n) := \frac{\sigma(n)}{n \log\log n} \qquad (n > 1)$$

satisfies

$$\limsup_{n\to\infty} G(n) = e^{\gamma} = 1.78107\ldots,$$

where γ is the Euler-Mascheroni constant.

Here γ is defined as the limit

$$\gamma := \lim_{n\to\infty} (H_n - \log n) = 0.57721\ldots,$$

where H_n denotes the *nth harmonic number*

$$H_n := \sum_{j=1}^{n} \frac{1}{j} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Gronwall's proof uses Mertens's theorem [5, Theorem 429], which says that if p denotes a prime, then

$$\lim_{x\to\infty} \frac{1}{\log x} \prod_{p\le x} \left(1 - \frac{1}{p}\right)^{-1} = e^{\gamma}.$$

Since $\sigma(n) > n$ for all $n > 1$, Gronwall's theorem "shows that the order of $\sigma(n)$ is always 'very nearly' n" (Hardy and Wright [5, p. 350]).

In 1915, the Indian mathematical genius Srinivasa Ramanujan proved an asymptotic inequality for Gronwall's function G, assuming the Riemann Hypothesis (RH). (Ramanujan's result was not published until much later [8]; for the interesting reasons, see [8, pp. 119–121] and [6, pp. 537–538].)

Theorem 2 (Ramanujan)**.** *If the Riemann Hypothesis is true, then*

$$G(n) < e^{\gamma} \qquad (n \gg 1).$$

Here $n \gg 1$ means for all sufficiently large n.

In 1984, the French mathematician Guy Robin [9] proved that a stronger statement about the function G is *equivalent* to the RH.

Theorem 3 (Robin)**.** *The Riemann Hypothesis is true if and only if*

$$G(n) < e^{\gamma} \qquad (n > 5040). \tag{1}$$

r	SA	Factorization	$\sigma(r)$	$\sigma(r)/r$	$G(r)$	$p(r)$	$G(11r)$
3		3	4	1.333	14.177		1.161
4	✓	2^2	7	1.750	5.357		1.434
5		5	6	1.200	2.521		0.943
6	✓	$2 \cdot 3$	12	2.000	3.429	2	1.522
8		2^3	15	1.875	2.561	2	1.364
9		3^2	13	1.444	1.834	3	1.033
10		$2 \cdot 5$	18	1.800	2.158	2	1.268
12	✓	$2^2 \cdot 3$	28	2.333	2.563	2	1.605
16		2^4	31	1.937	1.899	2	1.286
18		$2 \cdot 3^2$	39	2.166	2.041	3	1.419
20		$2^2 \cdot 5$	42	2.100	1.913	5	1.359
24	✓	$2^3 \cdot 3$	60	2.500	2.162	3	1.587
30		$2 \cdot 3 \cdot 5$	72	2.400	1.960	3	1.489
36	✓	$2^2 \cdot 3^2$	91	2.527	1.980	2	1.541
48	✓	$2^4 \cdot 3$	124	2.583	1.908	3	1.535
60	✓	$2^2 \cdot 3 \cdot 5$	168	2.800	1.986	5	1.632
72		$2^3 \cdot 3^2$	195	2.708	1.863	3	1.556
84		$2^2 \cdot 3 \cdot 7$	224	2.666	1.791	7	1.514
120	✓	$2^3 \cdot 3 \cdot 5$	360	3.000	1.915	2	1.659
180	✓	$2^2 \cdot 3^2 \cdot 5$	546	3.033	1.841	5	1.632
240	✓	$2^4 \cdot 3 \cdot 5$	744	3.100	1.822	5	1.638
360	✓	$2^3 \cdot 3^2 \cdot 5$	1170	3.250	1.833	5	1.676
720	✓	$2^4 \cdot 3^2 \cdot 5$	2418	3.358	1.782	3	1.669
840	✓	$2^3 \cdot 3 \cdot 5 \cdot 7$	2880	3.428	1.797	7	1.691
2520	✓	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	9360	3.714	1.804	7	1.742
5040	✓	$2^4 \cdot 3^2 \cdot 5 \cdot 7$	19344	3.838	1.790	2	1.751

Table 1: The set R of all known numbers r for which $G(r) \geq e^{\gamma}$. (Section 1 defines SA, $\sigma(r)$, and $G(r)$; Section 2 defines $p(r)$.)

The condition (1) is called *Robin's inequality.* Table 1 gives the known numbers r for which the reverse inequality $G(r) \geq e^\gamma$ holds, together with the value of $G(r)$ (truncated).

Robin's statement is elementary, and his theorem is beautiful and elegant, and is certainly quite an achievement.

In [9] Robin also proved, unconditionally, that

$$G(n) < e^\gamma + \frac{0.6483}{(\log\log n)^2} \qquad (n > 1). \tag{2}$$

This refines the inequality $\limsup_{n\to\infty} G(n) \leq e^\gamma$ from Gronwall's theorem.

In 2002, the American mathematician Jeffrey Lagarias [6] used Robin's theorem to give another elementary reformulation of the RH.

Theorem 4 (Lagarias)**.** *The Riemann Hypothesis is true if and only if*

$$\sigma(n) < H_n + \exp(H_n)\log(H_n) \qquad (n > 1).$$

Lagarias's theorem is also a beautiful, elegant, and remarkable achievement. It improves upon Robin's statement in that it does not require the condition $n > 5040$, which appears arbitrary. It also differs from Robin's statement in that it relies explicitly on the harmonic numbers H_n rather than on the constant γ.

Lagarias [7] also proved, unconditionally, that

$$\sigma(n) < H_n + 2\exp(H_n)\log(H_n) \qquad (n > 1).$$

The present note uses Robin's results to derive another reformulation of the RH. Before stating it, we give a definition and an example.

Definition 1. A positive integer N is *extraordinary* if N is composite and satisfies

(i). $G(N) \geq G(N/p)$ for all prime factors p of N, and

(ii). $G(N) \geq G(aN)$ for all multiples aN of N.

The smallest extraordinary number is $N = 4$. To show this, we first compute $G(4) = 5.357\ldots.$ Then as $G(2) < 0$, condition (i) holds, and since Robin's unconditional bound (2) implies

$$G(n) < e^\gamma + \frac{0.6483}{(\log\log 5)^2} = 4.643\ldots < G(4) \qquad (n \geq 5),$$

condition (ii) holds a fortiori.

No other extraordinary number is known, for a good reason.

Theorem 5. *The Riemann Hypothesis is true if and only if* 4 *is the only extraordinary number.*

This statement is elementary and involves prime numbers (via the definition of an extraordinary number) but not the constant γ or the harmonic numbers H_n, which are difficult to calculate and work with for large values of n. On the other hand, to disprove the RH using Robin's or Lagarias's statement would require only a *calculation* on a certain number n, while using ours would require a *proof* for a certain number N.

Here is a near miss. One can check that the number

$$\nu := 183783600 = 2^4 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \tag{3}$$

satisfies condition (i), that is, $G(\nu) \geq G(\nu/p)$ for $p = 2, 3, 5, 7, 11, 13, 17$. However, ν is not extraordinary, because $G(\nu) < G(19\nu)$. Thus 183783600 is not quite a counterexample to the RH!

In [8, Section 59] Ramanujan introduced the notion of a "generalized highly composite number." The terminology was changed to "superabundant number" by the Canadian-American mathematician Leonidas Alaoglu and the Hungarian mathematician Paul Erdős [2].

Definition 2 (Ramanujan and Alaoglu-Erdős)**.** A positive integer s is *superabundant (SA)* if

$$\frac{\sigma(n)}{n} < \frac{\sigma(s)}{s} \qquad (0 < n < s).$$

For example, the numbers $1, 2$, and 4 are SA, but 3 is not SA, because

$$\frac{\sigma(1)}{1} = 1 < \frac{\sigma(3)}{3} = \frac{4}{3} < \frac{\sigma(2)}{2} = \frac{3}{2} < \frac{\sigma(4)}{4} = \frac{7}{4}.$$

For lists of SA numbers, see the links at [10, Sequence A004394] and the last table in [2]. The known SA numbers s for which $G(s) \geq e^\gamma$ are indicated in the "SA" column of Table 1. Properties of SA numbers are given in [2, 3, 6, 8], Proposition 1, and Section 4.

As $\sigma(n)/n = G(n) \log\log n$, Gronwall's theorem yields $\limsup_{n\to\infty} \sigma(n)/n = \infty$, implying *there exist infinitely many SA numbers.*

Let us compare Definition 2 with condition (i) in Definition 1. If $n < s$, then $\sigma(n)/n < \sigma(s)/s$ is a weaker inequality than $G(n) < G(s)$. On the other hand, condition (i) only requires $G(n) \leq G(N)$ for factors $n = N/p$, while Definition 2 requires $\sigma(n)/n < \sigma(s)/s$ for all $n < s$. In particular, *the near miss* (3) *is the smallest SA number greater than* 4 *that satisfies* (i). For more on (i), see Section 5.

Amir Akbary and Zachary Friggstad [1] observed that, "*If there is any counterexample to Robin's inequality, then the least such counterexample is a superabundant number.*" Combined with Robin's theorem, their result implies *the RH is true if and only if $G(s) < e^\gamma$ for all SA numbers $s > 5040$.*

Here is an analog for extraordinary numbers of Akbary and Friggstad's observation on SA numbers.

Corollary 1. *If there is any counterexample to Robin's inequality, then the maximum* $M := \max\{G(n) : n > 5040\}$ *exists and the least number* $N > 5040$ *with* $G(N) = M$ *is extraordinary.*

Using Gronwall's theorem and results of Alaoglu and Erdős, we prove two properties of SA numbers.

Proposition 1. *Let* S *denote the set of superabundant numbers.*
SA1. *We have*

$$\limsup_{s \in S} G(s) = e^{\gamma}.$$

SA2. *For any fixed positive integer* n_0*, every sufficiently large number* $s \in S$ *is a multiple of* n_0*.*

The rest of the paper is organized as follows. The next section contains three lemmas about the function G; an alternate proof of the first uses Proposition 1. The lemmas are used in the proof of Theorem 5 and Corollary 1, which is in Section 3. Proposition 1 is proved in Section 4. Section 5 gives some first results about numbers satisfying condition (i) in Definition 1.

We intend to return to the last subject in another paper, in which we will also study numbers satisfying condition (ii).

2. Three lemmas on the function G

The proof of Theorem 5 requires three lemmas. Their proofs are unconditional.

The first lemma generalizes Gronwall's theorem (the case $n_0 = 1$).

Lemma 1. *If* n_0 *is any fixed positive integer, then* $\limsup_{a \to \infty} G(an_0) = e^{\gamma}$.

We give two proofs.

Proof 1. Theorem 1 implies $\limsup_{a\to\infty} G(an_0) \le e^{\gamma}$. The reverse inequality can be proved by adapting that part of the proof of Theorem 1 in [5, Section 22.9]. Details are omitted. □

Proof 2. The lemma follows immediately from Proposition 1. □

The remaining two lemmas give properties of the set R of all known numbers r for which $G(r) \ge e^{\gamma}$.

Lemma 2. *Let* R *denote the set*

$$R := \{r \le 5040 : G(r) \ge e^{\gamma}\}.$$

If $r \in R$ *and* $r > 5$*, then* $G(r) < G(r/p)$*, for some prime factor* p *of* r*.*

Proof. The numbers $r \in R$ and the values $G(r)$ are computed in Table 1. Assuming $G(r) < G(r/p)$ for some prime factor p of r, denote the smallest such prime by

$$p(r) := \min\{\text{prime } p \mid r : G(r/p) > G(r)\}.$$

Whenever $5 < r \in R$, a value of $p(r)$ is exhibited in the "$p(r)$" column of Table 1. This proves the lemma. $\square$

Lemma 3. *If $r \in R$ and $p \geq 11$ is prime, then $G(pr) < e^{\gamma}$.*

Proof. Note that if $p > q$ are odd primes not dividing a number n, then

$$G(pn) = \frac{\sigma(pn)}{pn \log\log pn} = \frac{p+1}{p} \frac{\sigma(n)}{n \log\log pn} < \frac{q+1}{q} \frac{\sigma(n)}{n \log\log qn} = G(qn).$$

Also, Table 1 shows that no prime $p \geq 11$ divides any number $r \in R$, and that $G(11r) < 1.76$ for all $r \in R$. As $1.76 < e^{\gamma}$, we obtain $G(pr) \leq G(11r) < e^{\gamma}$. $\square$

Note that the inequality $G(pn) < G(qn)$ and its proof remain valid for *all* primes $p > q$ not dividing n, if $n > 1$, since then $\log\log qn \neq \log\log 2 < 0$ when $q = 2$.

3. Proof of Theorem 5 and Corollary 1

We can now prove that our statement is equivalent to the RH.

Proof of Theorem 5 and Corollary 1. Assume $N \neq 4$ is an extraordinary number. Then condition (ii) and Lemma 1 imply $G(N) \geq e^{\gamma}$. Thus if $N \leq 5040$, then $N \in R$, but now since $N \neq 4$ is composite we have $N > 5$, and Lemma 2 contradicts condition (i). Hence $N > 5040$, and by Theorem 3 the RH is false.

Conversely, suppose the RH is false. Then from Theorems 1 and 3 we infer that the maximum

$$M := \max\{G(n) : n > 5040\} \tag{4}$$

exists and that $M \geq e^{\gamma}$. Set

$$N := \min\{n > 5040 : G(n) = M\} \tag{5}$$

and note that $G(N) = M \geq e^{\gamma}$. We show that N is an extraordinary number.

First of all, N is composite, because if N is prime, then $\sigma(N) = 1 + N$ and $N > 5040$ imply $G(N) < 5041/(5040 \log\log 5040) = 0.46672\ldots$, contradicting $G(N) \geq e^{\gamma}$.

Since (4) and (5) imply $G(N) \geq G(n)$ for all $n \geq N$, condition (ii) holds. To see that (i) also holds, let prime p divide N and set $r := N/p$. In the case $r > 5040$, as $r < N$ the minimality of N implies $G(N) > G(r)$. Now consider the case $r \leq 5040$.

By computation, $G(n) < e^\gamma$ if $5041 \le n \le 35280$, so that $N > 35280 = 7 \cdot 5040$ and hence $p \ge 11$. Now if $G(r) \ge e^\gamma$, implying $r \in R$, then Lemma 3 yields $e^\gamma > G(pr) = G(N)$, contradicting $G(N) \ge e^\gamma$. Hence $G(r) < e^\gamma \le G(N)$. Thus in both cases $G(N) > G(r) = G(N/p)$, and so (i) holds. Therefore, $N \ne 4$ is extraordinary. This proves both the theorem and the corollary. $\square$

Remark 1. The proof shows that Theorem 5 and Corollary 1 remain valid if we replace the inequality in Definition 1 (i) with the strict inequality $G(N) > G(N/p)$.

4. Proof of Proposition 1

We prove the two parts of Proposition 1 separately.

Proof of SA1. It suffices to construct a sequence $s_1, s_2, \ldots \to \infty$ with $s_k \in S$ and $\limsup_{k\to\infty} G(s_k) \ge e^\gamma$. By Theorem 1, there exist positive integers $\nu_1 < \nu_2 < \cdots$ with $\lim_{k\to\infty} G(\nu_k) = e^\gamma$. If $\nu_k \in S$, set $s_k := \nu_k$. Now assume $\nu_k \notin S$, and set $s_k := \max\{s \in S : s < \nu_k\}$. Then $\{s_k + 1, s_k + 2, \ldots, \nu_k\} \cap S = \emptyset$, and we deduce that there exists a number $r_k \le s_k$ with $\sigma(r_k)/r_k \ge \sigma(\nu_k)/\nu_k$. As $s_k \in S$, we obtain $\sigma(s_k)/s_k \ge \sigma(\nu_k)/\nu_k$, implying $G(s_k) > G(\nu_k)$. Now since $\lim_{k\to\infty} \nu_k = \infty$ and $\#S = \infty$ imply $\lim_{k\to\infty} s_k = \infty$, we get $\limsup_{k\to\infty} G(s_k) \ge e^\gamma$, as desired. $\square$

Proof of SA2. We use the following three properties of a number $s \in S$, proved by Alaoglu and Erdős [2].

AE1. *The exponents in the prime factorization of s are non-increasing, that is, $s = 2^{k_2} \cdot 3^{k_3} \cdot 5^{k_5} \cdots p^{k_p}$ with $k_2 \ge k_3 \ge k_5 \ge \cdots \ge k_p$.*
AE2. *If $q < r$ are prime factors of s, then $|\lfloor k_q \frac{\log q}{\log r} \rfloor - k_r| \le 1$.*
AE3. *If q is any prime factor of s, then $q^{k_q} < 2^{k_2+2}$.*

To prove SA2, fix an integer $n_0 > 1$. Let K denote the largest exponent in the prime factorization of n_0, and set $P := P(n_0)$, where $P(n)$ *denotes the largest prime factor of n.* As n_0 divides $(2 \cdot 3 \cdot 5 \cdots P)^K$, by AE1 it suffices to show that the set

$$F := \{s \in S : s \text{ is not divisible by } P^K\} = \{s \in S : 0 \le k_P = k_P(s) < K\}$$

is finite.

From AE2 with $q = 2$ and $r = P$, we infer that $k_2 = k_2(s)$ is bounded, say $k_2(s) < B$, for all $s \in F$. Now if q is any prime factor of s, then AE1 implies $k_q = k_q(s) < B$, and AE3 implies $q^{k_q} < 2^{B+2}$. The latter with $q = P(s)$ forces $P(s) < 2^{B+2}$. Therefore, $s < (2^{B+2}!)^B$ for all $s \in F$, and so F is a finite set. $\square$

Remark 2. We outline another proof of SA2. Observe first that, if p^{k+1} does not divide n, then (compare the proof of [5, Theorem 329])

$$\frac{\sigma(n)}{n} \le \frac{n}{\varphi(n)}\left(1 - \frac{1}{p^{k+1}}\right),$$

where $\varphi(n)$ is Euler's totient function. Together with the classical result [5, Theorem 328]

$$\limsup_{n\to\infty} \frac{n}{\varphi(n)\log\log n} = e^{\gamma},$$

this implies that there exists $\epsilon = \epsilon(n_0) > 0$ such that, if $n \gg 1$ is not multiple of n_0, then

$$G(n) \le e^{\gamma} - \epsilon,$$

so that, by SA1, n cannot be SA.

5. GA numbers of the first kind

Let us say that a positive integer n is a *GA number of the first kind* (*GA1 number*) if n is composite and satisfies condition (i) in Definition 1 with N replaced by n, that is, $G(n) \ge G(n/p)$ for all primes p dividing n. For example, 4 is GA1, as are all other extraordinary numbers, if any. Also, the near miss 183783600 is a GA number of the first kind. By Lemma 2, if $4 \ne r \in R$, then r is not a GA1 number.

Writing $p^k \parallel n$ when $p^k \mid n$ but $p^{k+1} \nmid n$, we have the following criterion for GA1 numbers.

Proposition 2. *A composite number n is a GA number of the first kind if and only if prime $p \mid n$ implies*

$$\frac{\log\log n}{\log\log \frac{n}{p}} \le \frac{p^{k+1}-1}{p^{k+1}-p} \qquad (p^k \parallel n).$$

Proof. This follows easily from the definitions of GA1 and $G(n)$ and the formulas

$$\sigma(n) = \prod_{p^k \parallel n} (1 + p + p^2 + \cdots + p^k) = \prod_{p^k \parallel n} \frac{p^{k+1}-1}{p-1}. \qquad \square$$

The next two propositions determine all GA1 numbers with exactly two prime factors.

Proposition 3. *Let p be a prime. Then $2p$ is a GA number of the first kind if and only if $p = 2$ or $p > 5$.*

Proof. As $G(2) < 0 < G(2p)$, the number $2p$ is GA1 if and only if $G(2p) \geq G(p)$. Thus $2p$ is GA1 for $p = 2$, but, by computation, not for $p = 3$ and 5. If $p > 5$, then since $3 \log\log x > 2 \log\log 2x$ for $x \geq 7$, we have

$$\frac{G(2p)}{G(p)} = \frac{\sigma(2p)}{2p \log\log 2p} \div \frac{\sigma(p)}{p \log\log p} = \frac{3(p+1)}{2p \log\log 2p} \cdot \frac{p \log\log p}{p+1} = \frac{3 \log\log p}{2 \log\log 2p} > 1.$$

Thus $2p$ is GA1 for $p = 7, 11, 13, \ldots$. This prove the proposition. $\square$

Proposition 4. *Let p and q be odd primes. Then pq is not a GA number of the first kind.*

Proof. As $(x+1) \log\log y < x \log\log xy$ when $x \geq y \geq 3$, it follows that if $p > q \geq 3$ are primes, then

$$\frac{G(pq)}{G(q)} = \frac{(p+1)(q+1)}{pq \log\log pq} \div \frac{q+1}{q \log\log q} = \frac{(p+1) \log\log q}{p \log\log pq} < 1,$$

and if $p \geq 3$ is prime, then

$$\frac{G(p^2)}{G(p)} = \frac{p^2+p+1}{p^2 \log\log p^2} \div \frac{p+1}{p \log\log p} = \frac{(p^2+p+1) \log\log p}{(p^2+p) \log\log p^2} < 1.$$

Hence pq is not GA1 for odd primes p and q. $\square$

6. Concluding remarks

Our reformulation of the RH, like Lagarias's, is attractive because the constant e^γ does not appear. Also, there is an elegant symmetry to the pair of conditions (i) and (ii): the value of the function G at the number N is not less than its values at the quotients N/p and at the multiples aN. The statement reformulates the Riemann Hypothesis in purely elementary terms of divisors, prime factors, multiples, and logarithms.

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